CERTAIN PROBLEMS IN VARIATIONAL INEQUALITIES

DISSERTATION
SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy
IN
MATHEMATICS

BY
MOHAMMAD FIRDOSH KHAN

Under the Supervision of
PROF. ABUL HASAN SIDDIQI
(Pro-Vice-Chancellor, A. M. U., Aligarh)

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)
1994

Dedicated To My Parents

ALIGARH MUSLIM UNIVERSITY
ALIGARH

Prof. Abul Hasan Siddiqi
PRO-VICE-CHANCELLOR

CERTIFICATE

Certified that Mr. Mohammad Firdosh Khan has carried out the work on 'Certain Problems in Variational Inequalities' under my supervision and the work is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.

(Prof. Abul Hasan Siddiqi)
Supervisor

Chairman
Deptt. of Mathematics
A.M.U., Aligarh

TEL. (0571) OFF. : 401727, 400297, RES. : 401202, INT. 205, FAX : 571-400827, TELEX : 564-230 AMU IN
E-MAIL : AHSPVC @ AMU. ERNET. IN ALIGARH-202002 (INDIA)

ACKNOWLEDGEMENT

It is a great fortune and a matter of pride and privilage for me to have the brilliant supervision of Prof A.H. Siddiqi. Pro-Vice-Chancellor, Aligarh Muslim University, Aligarh. I gratefully acknowledge his able guidance, encouragement and cooperation through out this work. His constructive criticism and valuable suggestions have been the source of inspiration for me and created interest in further studies.

It is great pleasure for me to express my gratitude to Prof. M.Z. Khan, Chairman, Deptt. of Mathematics, A.M.U. Aligarh for providing me the necessary facilities in the department in carrying out this work and to Dr. (Mrs) Azra Hasan Siddiqi for her encouragement and interest during the course of this work.

My maternal uncle, Mr. Mohd. Haroon Khan has always been a brilliant source of inspiration and courage for me since the very begining of my studies. I do not find words to express indebtedness to him for all sorts of help and support.

I am over whelmed with the sence of indebtedness to my teacher and colleague Dr. Qamrul Ha_san Ansari whose scholarly and encouraging help rendered to me in the completion of this work.

Thanks are also due to my colleagues and friends for their cooperation. I feel pleasure to acknowledge Dr. Rais Ahmad, Dr. Shamshad Husain. Dr. Abdul Khaliq, Dr. Kaleem Raza Kazmi; Mr. Kaleemuddin Ahmad, Mr. Ayaz Ahmad, Mr. Mohd. Ali. Mr. Zaki Anwar, for their help and cooperation.

Hackneyed words are unable to express my heart felt thanks to Mrs. Shakeba for her wholehearted coopration, moral support, and help.

I would be an injustic, if I fail to express my thanks to my younger brother and sister who are also my well wisher and desirous of my success. Thanks are also due to Mr Badre' Alam for his moral support during my studies in Aligarh.

The University fellowship (J.R.F) awarded by Aligarh Muslim University Aligarh to carry out this work is also thankfully acknowledged.

Careful typing by Mr. Fazal Hasnain Naqvi and Mr. Ali Abbas Naqvi is thankfully acknowledged.

(Mohammad Firdosh Khan)

PREFACE

Variational inequalities are mathematical models, in terms of operators of different physical phinomena. This topic has been originated from the study of Signorini problems by Fichera [28] and Stampacchia [91], separately. This has been developed recently by some well known mathematicians. Application of variational inequalities to variety of problems are very impressive and have great contribution to the richness of this field. Moreover some of these developments have been employed in other areas of mathematical and engineering sciences including elasticity, transportation and economics, equilibrium theory, nonlinear programming and operations research. Now a vast literature is available on the theory of linear variational inequalities and their applications to different areas but numerical solution of variational inequalities in terms of nonlinear operators is still an unexplored field.

The main objective of this work is to present a brief introduction to various types of variational inequalities and their existence theory in different spaces.

There are five chapters in this dissertation. Chapter I, contains basic definations and notations which have been used in the subsequent chapters. In chapter II, the general error estimates for variational and general variational inequalities are discussed and an application of error estimation for variational inequality is also presented. chapter III, deals with the Wiener-Hopf equations and variational inequalities. The equivalence between these two are also studied and convergence of approximate solution to the exact solution are discussed. In chapter IV, The existence theory of variational - like inequalities in Euclidean space, reflexive Banach space and Hausdorff topological vector space has been discussed. Existence without convexity has been studied. The existence theory of vector variational and general vector variational inequalities in different spaces is presented in chapter V.

CONTENTS

--------------****************-------------

-----------********---

CHAPTER-I

If $u_n \to u$ strongly in K and (1.2.7) holds, F is said to be lower semicontinuous. Similarly upper semicontinuous can be defined

DEFINITION 1.2.3. A real valued function F defined on a convex subset K of X is said to be quasi-concave if for every real number $t > 0$, the set $\{u \in K : F(u) > t\}$ is convex, F is said to be quasi-convex if $-F$ is quasi-concave.

DEFINITION 1.2.4. A subset K of R^n is said to be contractible if there is a point $u \in K$ and a continuous function $g : K \times [0,1] \to K$ is such that $g(v,o) = v$ and $g(v,1) = u$, for all $v \in K$.

Remark 1.2.1. Every convex set is contractible but converse is not true.

DEFINITION 1.2.5. [74]. Let X be a real linear space. A nonempty subset C of X is called

(I) Cone if

$C + C = C$ and $\lambda C \subset C$, for any $\lambda > 0$, $\lambda \in R$

(II) a pointed cone if

C is a cone and $C \cap (-C) = \{0\}$

DEFINITION 1.2.6. A polar cone is denoted by C^* and defined as

$$C^* = \{u^* \in X^* : (u^*, u) \geq 0, \text{ for every } u \in C\}$$

DEFINITION 1.2.7. A mapping $T : H \to H^*$ is called

(I) monotone, if

$$\langle T(u) - T(v), u-v \rangle \geq 0, \text{ for all } u,v \in K,$$

(II) strictly monotone, if

$$\langle T(u) - T(v), u-v \rangle = 0, \text{ iff } u = v, \text{ for all } u,v \in H$$

(III) antimonotone, if

$$\langle T(u) - T(v), u-v \rangle \leq 0, \text{ for all } u,v \in H.$$

(IV) lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$||T(u)-T(v)|| \leq \gamma ||u-v||^2, \text{ for all } u,v \in H$$

(V) hemicontinuous, if for every $u,v \in H$, the map

$$t \to \langle T(u+tv), v \rangle \text{ is continuous at } 0^+$$

(VI) α-strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle T(u)-T(v), u-v \rangle \geq \alpha ||u-v||^2, \ \forall \ u,v \in H$$

DEFINITION 1.2.8. [8]. A multivalued mapping $F : X \to 2^X$ is called the KKM-mapping if for every finite subset $\{u_1, u_2, \ldots, u_n\}$ of X, $\mathrm{Conv}\,\{u_1, u_2, \ldots, u_n\} \subset \bigcup_{i=1}^{n} F(u_i)$.

The following results are used in Chapter 4 and 5 to prove the results.

THEOREM (Ky Fan) 1.2.1. [25]. Let K be a non empty set in a Hausdorff topological vector space X. Let $F : K \to 2^X$ be KKM-mapping. If $F(u)$ is closed for all $u \in K$ and is compact for at least one $u \in K$, then $\bigcap_{u \in K} F(u) \neq \phi$.

THEOREM 1.2.2. [85]. Let E be a nonempty compact convex set in a Hausdorff topological vector space.

Let A be a subset of $E \times E$ having the following properties:

(1) for every $u \in E$, $(u,u) \in A$

(2) for each fixed $u \in E$ the set $A_u = \{v \in E : (u,v) \in A\}$ is closed is E,

(3) for each fixed $v \in E$ the set $A_v = \{u \in E : (u,v) \notin A\}$ is closed in E,

Then there exist a point $v_0 \in E$ such that $E \times \{v_0\} \subset A$.

The following definition is suggested by a recent generalization of KKM-theorem obtained by C. Horvath [37].

DEFINITION 1.2.9. [37]. Let X be a topological space and let $\{\Gamma_A\}$ be a given family of nonempty and contractible subsets of X, indexed by finite subset of X.

(I) A pair $(X,\{\Gamma_A\})$ is said to be an H-space if and only if $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

(II) A subset $D \subset X$ is called H-convex if and only if $\Gamma_A \subset D$ hold for every finite subset $A \subset D$.

(III) A subset $D \subset X$, is called weakly H-convex if and only if $\Gamma_A \cap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to showing that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.

(IV) A subset $K \subset X$ is called H-compact if and only if there exist a compact and weakly H-convex set $D \subset X$ such that $K \cup A \subset D$ for every finite subset $A \subset X$.

DEFINITION 1.2.10. [8]. A multivalued mapping $F : X \to 2^X$ is called H-KKM mapping if $\Gamma_A \subset \bigcup_{u \in A} F(u)$ for every finite subset $A \subset X$.

THEOREM 1.2.3. [8]. Let $(X,\{\Gamma_A\})$ be an H-space and $F:X \to 2^X$ be an H-KKM multivalued map such that

(a) for each $u \in E$, $F(x)$ is compactly closed, that is $B \cap F(u)$ is closed in B, for every compact set $B \subset X$

(b) there exist a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for each weakly H-convex set D with $K \subset D \subset X$ we have $\bigcap_{u \in D} (F(u) \cap D) \subset L$

Then

$$\bigcap_{u \in X} F(u) \neq \phi.$$

1.3. SOBOLEV SPACES

DEFINITION 1.3.1. [74]. The Sobolev space of order 1 on Ω (an open subset of R^N), denoted by $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \{f \in L_2(\Omega): \frac{\partial f}{\partial x_i} \in L_2(\Omega),\ 1 \leq i \leq N\} \qquad (1.3.1)$$

where $\frac{\partial f}{\partial x_i}$ are derivative in the sence of distribution.

REMARK 1.3.1. (i) If $n = 1$, $\Omega = [a,b]$, $f : [a,b] \to R$ then

$$H^1(a,b) = \{f \in L_2(a,b): \frac{\partial f}{\partial x_i} \in L_2(a,b)\} \qquad (1.3.2)$$

(ii) $f \in L_2(\Omega)$ need not imply $\frac{\partial f}{\partial x_i} \in L_2(\Omega)$

for example,

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x > 0 \end{cases}, \quad \Omega = [-1,1]$$

$f \in L_2(\Omega)$, but $\frac{\partial f}{\partial x} = \phi(0) \notin L_2(\Omega)$

THEOREM 1.3.1. [74]. $H^1(\Omega)$ is a Hilbert space with respect to the inner product,

$$\langle f,g\rangle_{1,L_2(\Omega)} = \langle f,g\rangle_{L_2(\Omega)} + \sum_{i=1}^{N} \langle \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}\rangle_{L_2(\Omega)} \tag{1.3.3}$$

REMARK 1.3.2. If $\Omega = [a,b]$, then $H^1(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx + \int_a^b \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\,dx. \tag{1.3.4}$$

DEFINITION 1.3.2. [74]. For any integer m and $1 \leq p \leq \infty$ the Sobolev space (generalized) is the space of all functions $f \in L_p(\Omega)$ whose all derivatives in the sense of distribution of order $\leq m$ also belongs to $L_p(\Omega)$, that is

$$W^{m,p}(\Omega) = \{f \in L_p(\Omega) : D^\alpha f \in L_p(\Omega), \text{ for all } \alpha, |\alpha| \leq m\} \tag{1.3.5}$$

$W^{m,p}(\Omega)$ is a Banach space with the norm,

$$||f||_{W^{m,p}} = [\sum_{|\alpha|\leq m} ||D^\alpha f||^p_{L_p(\Omega)}]^{1/p} \tag{1.3.6}$$

For $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$, $H^m(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f,g\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle D^\alpha f,\ D^\alpha g\rangle_{L_2(\Omega)}$$

$$= \int_\Omega [\sum_{|\alpha|\leq m} D^\alpha(f)\ D^\alpha(g)]\ dx \qquad (1.3.7)$$

The norm induced by this inner product is

$$||f||_{H^m(\Omega)} = [\langle f,f\rangle_{H^m(\Omega)}]^{\frac{1}{2}} \qquad (1.3.8)$$

$H^m(\Omega)$ is called a Sobolev space of order m.

$$H_o^m(\Omega) = \{f : f \in H^m(\Omega) \text{ and } \frac{\partial^k f}{\partial n^k}\Big|_\Gamma = 0,\ 0 \leq k \leq n-1\}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary Γ.

1.4. THE FINITE ELEMENT METHOD:

The finite element is concerned with the construction of finite dimensional subspace of functional spaces specially Sobolev spaces and study of abstract variational problems and in general, Variational Inequalities on such subspaces. This reduces BVPs to the study of matrix equations and inequalities which can be handled through computer. Structural engineers like Argyris, Turner, Clough, Martin and Top have been using the finite element method in structural analysis since 1954. However a mathematical breakthrough came in the paper of Zlamal in 1968 and a sound mathematical theory of this topic has been developed in the last two decades.

Let V_h be the finite dimensional subspace of a given Hilbert space V. Then finding $u_h \varepsilon V_h$ such that

$$a(u_h, v_h) = f(v_h), \text{ for all } v_h \varepsilon V_h \tag{1.4.1}$$

is known as approximate variational problem. Both the problems (1.2.1) and (1.4.1) possess unique solutions if $a(.,.)$ is bilinear, continuous and coercive [74]

THEOREM 1.5.1. [74]. (Cea's Lemma). Let u and u_h denote the solutions of (1.2.1) and (1.4.1) respectively, there exists a constant C independent of the subspace V_h such that

$$||u-u_h|| \leq C \inf ||u-v_h|| \text{ for all } v_h \varepsilon V_h \tag{1.4.2}$$

REMARK 1.4.1. The inequality (1.4.2) shows that the problem of estimating the error $||u-u_h||$ is reduced to a problem in approximation theory namely, the evaluation of the distance $d(u,V_h) = \text{Inf } ||u-V_h||$ between $u \varepsilon V$ and the subspace V_h of

COROLLARY 1.4.1. Suppose there exist a dense subspace U of V and a mapping $\gamma_h : U \to V$ such that $\lim_{h \to o} ||v - \gamma_h v|| = 0$ for all $v \in U$. Then

$$\lim_{h \to o} ||u - u_h|| = 0 \tag{1.4.3}$$

DEFINITION 1.4.1. Let $\Omega \subset R^2$ be a polygonal domain. Then a finite collection of triangles T_h is called triagulation if the following conditions are satisfied.

(i) $\bar{\Omega} = \bigcup_{k \in T_h} \bar{K}$, $\bar{K}$ denotes a triangulation with boundary.

(ii) $K_1 \cap K_2 = \phi$ for $K_1, K_2 \in T_h$, $K_1 \neq K_2$

(iii) $\bar{K}_1 \cap \bar{K}_2$ = a vertex or a side, that is, if we consider two triangles, their boundaries may have one vertex common or one side common.

REMARK 1.4.2. Let $P(K)$ be a function space defined on $K \in T_h$ such that $P(K) \subset H^1(K)$. Generally, $P(K)$ will be space of some degree.

THEOREM 1.4.2.[74]. Let $C^{\circ}(\Omega)$ be the space of continuous real-valued function on $\bar{\Omega}$ and

$$V_h = \{v_h \in C^\circ(\bar{\Omega}) : v_h\big|_k \in P(K),\ K \in T_h\} \qquad (1.4.4)$$

where $V_h\big|_k$ denotes the restriction of v_h on K and $P(K) \subset H^1(K)$; then $V_h \subset H^1(\Omega)$.

REMARK 1.4.3. Let $h = \max_{k \in T_h}$ (diameter of K), $N(h)$ = the number of nodes of the triangulation, $P(K) = P_1(K)$ = space of polynomials of degree less than or equal to 1 in x and y,

$$V_h = \{v_h : V_h\big|_k \in P_1(K),\ k \in T_h\} \qquad (1.4.5)$$

(i) It can be seen that $V_h \subset C^\circ(\bar{\Omega})$.

(ii) The functions w_i, $i = 1,2,\ldots\ldots N(h)$, defined by

$$w_i = \begin{cases} 1 & \text{at the ith node} \\ 0 & \text{at other nodes} \end{cases}$$

from a basis of V_h.

(iii) In view of (ii) and theorem (1.4.2), V_h defined in this remark is a subspace of $H^1(\Omega)$ of dimension $N(h)$.

THEOREM 1.4.3.[74]. (Convergence Theorem). Let $K_h \subset K$ and there exists $U \subset V$ dense in K and $\gamma_h : U \to K_h$ such that

$$\lim_{h \to o} ||\gamma_h v - v||_V = 0 \quad \text{for all} \quad v \in U \qquad (1.4.6)$$

Then

$$\lim_{h \to o} ||u-u_h||_V = 0 \tag{1.4.7}$$

where u and u_h are solutions of (1.2.1) and (1.4.1) respectively.

CHAPTER-II

VARIATIONAL INEQUALITIES AND ITS ERROR ESTIMATES

2.1. INTRODUCTION

Variational inequalities are now fundamental tool in studying the obstacle and unilateral problems arising in engineering and mathematical sciences. Finite element techniques are being applied for obtaining numerical solutions of variational inequalities. Using the piecewise linear elements, Falk [24], Mosco and Strang [59], Brezzi, Hagger and Raviart [11] have shown the $O(h)$ convergence in the energy norm.

Using the unilateral approximation result of Mosco and Strang [59], Noor [61], obtained the error estimates for the finite element approximation of a class of variational inequalities in the L_2-norm, which are of order h^2. Siddiqi [77] derive a general abstract error estimate for general variational inequality.

In this chapter we discussed a method for obtaining general error estimates for the approximation of a class of variational inequalities. In Section 2.2 of this chapter we consider an approximation scheme and establish the general error estimate. In Section 2.3, we studied the general error estimate for general variational inequality. In the last section, we discuss an application and find its error estimate by using finite element method.

2.2. APPROXIMATION SCHEME AND GENERAL ERROR ESTIMATE FOR VARIATIONAL INEQUALITY.

Let K be a closed convex subset of H and $f \in H'$ we then consider approximation of problems of following type:

PROBLEM 2.2.1. Find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u), \forall v \in K \tag{2.2.1}$$

This problem is introduced by G. Stampacchia. In 1967 Lions and Stampacchia proved the first theorem of this field.

THEOREM 2.2.1 [48]. If $a(.,.) : H \times H \to R$ is a continuous an coercive bilinear form on H, then problem (2.2.1) has a solution.

Let V_h be finite dimensional subspace of H $\{w_i\}_{i=1}^{q}$ is a basis of V_h. Then any arbitrary element $v_h \in V_h$ can be written as

$$v_h = \sum_{i=1}^{q} \beta_i w_i, \quad \text{where } \beta_i \in R$$

Then we construct a closed convex subset K_h of V_h such that the following conditions are satisfied

(I) K_h should reduce to a finite number of constraints on the β

(II) K_h should be a good approximation to K.

The approximate problem is as follows:

PROBLEM 2.2.2
$$\begin{cases} \text{Find } u_h \in K_h \text{ such that} \\ a(u_h, v_h - u_h) \geq (f, v_h - u_h), \ \forall\, v_h \in K_h. \end{cases} \qquad (2.2.2)$$

and we can show that this approximation problem has a unique solution by using Lions and Stampacchia theorem (2.2.1)

Suppose that W is a Hilbert space dense in H' and the injection W in to H' is continuous. Then there exist a continuous injection $i : H \to W'$ such that $i(H)$ is dense in W' and

$$(i(v), w)_{W,W'} = (v,w)_{H,H'} \ \forall\, v \in H,\ w \in W$$

we will henceforth identify H with a subspace of W dense in W' and pairing between H and H' and W and W' is denoted by $(.,.)$.

Now we state the following general error estimate.

THEOREM 2.2.1. [24]. Let u and u_h be the solution of problem (2.2.1) and (2.2.2), respectively $A \in L(H,H')$ the map defined, for $u \in H$, by $a(u,v) = (Au,v)$, $\forall\, v \in H$. Finally suppose that $f - Au \in W$. Then

$$||u-u_h|| \leq \{\frac{\beta^2}{\alpha^2}||u-v_h||_H^2 + \frac{2}{\alpha}\,||f-Au||_W\,[\,||u-v_h||_{W'} + ||u_h-v||_{W'}]\}^{\frac{1}{2}} \qquad (2.2.3)$$

2.3. GENERAL ERROR ESTIMATE IN GENERAL VARIATIONAL INEQUALITY

Let K be a closed convex set in H. Given $T,A,g:H \to H'$ the nonlinear continuous operators, we consider the following problem:

PROBLEM 2.3.1. [77]. Find $u \in H$ such that

$$\langle Tu,\ g(v)-g(u)\rangle \geq \langle Au,\ g(v)-g(u)\rangle \quad \text{for all } g(u),g(v) \in K, \qquad (2.3.$$

We present here a general estimate for the general variational inequality (2.3.1) which contains the result of Falk [24] as a special case. They hold for any finite dimensional subspace K and approximate constraint set K_h and represent a significant improvement of all the estimates for corresponding elliptic variational inequalities. We shall assume that there exist a Hilbert space W which is densely and continuously imbeded in the dual space H'. It is possible to identify H with a subspace of W', that is dense in W' by a continuous injection. Let $K \subset H$ be a finite dimensional subspace and $K_h \subset H$ *be a finite dimensional convex set. An* approximation of (2.3.1) is to find $u_h \in K_h$ such that

$$<Tu, g(v_h)-g(u_h)> \geq <A(u_h), g(v_h)-g(u_h)> \qquad (2.3.2)$$

with these hypothesis and preliminaries, we have the following error estimate.

THEOREM 2.3.1. [77]. Let $u \in K$ and $u_h \in K_h$ be the solution of (2.3.1) and (2.3.2) respectively. Let T and g be strongly monotone and lipschitz continuous operator. If A is a lipschitz continuous antimonotone operator, and $Tu-Au \in W$, then there exist a constant $c > 0$, such that

(i) For $K_h \not\subset K$,

$$||u-u_h|| \leq C\{||u-v_h||_H^2 + ||v-u_h||_H^2 + ||v_h-v||_H^2 +$$

$$+(||A(u)-T(u)||_W + ||u-v_h||_W + ||A(u_h)-T(u_h)||_W ||u_h-v||_{W'})\}^{\frac{1}{2}}$$

$$\text{for all } v \in K,\ v_h \in K_h \qquad (2.3.3)$$

(ii) $K_h \subset K$,

$$||u-u_h|| < \{||u-v_h||_H^2 + (||A(u)-T(u)||_W ||u-v_h||_{W'})\}^{\frac{1}{2}}$$

$$\text{for all } v_h \in K_h \tag{2.3.4}$$

PROOF. Since $u \in K$ and $u_h \in K_h$ are solutions of (2.3.1) and (2.3.2) respectively so by adding these inequalities we get

$$<Tu,g(v)-g(u)>+<Tu_h,g(v_h)-g(u_h)> \geq <Au,g(v)-g(u)> + <Au_h,g(v_h)-g(u_h)>$$

or $$<Tu,g(v)>+<Tu_h,g(v_h)>+<Au,g(u)-g(v)>+<Au_h,g(u_h)-g(v_h)> \geq <Tu,g(u)>+<Tu_h,g(u_h)>$$

or $$<Tu,g(v)>+<Tu_h,g(v_h)>+<Au,g(u)-g(v)>+<Au_h,g(u_h)-g(v_h)> -\{<Tu,g(u_h)>+<Tu_h,g(u)>\} \geq -\{<Tu,g(u_h)>+<Tu_h,g(u)>\}+ +<Tu,g(u)>+<Tu_h,g(u_h)>$$

or $$<Tu,g(v)-g(u_h)>+<Tu_h,g(v_h)-g(u)>+<Au,g(u)-g(v)> + +<Au_h,g(u_h)-g(v_h)> \geq <Tu-Tu_h,g(u)-g(u_h)> \tag{2.3.5}$$

l.h.s of (2.3.5)

$$= \langle Tu-Tu_h, g(v)-g(u_h)\rangle - \langle Tu_h, g(u_h)-g(v)\rangle$$

$$+\langle Tu-Tu_h,\ g(u)-g(v_h)\rangle - \langle Tu, g(u)-g(v_h)\rangle$$

$$+\langle Au, g(u)-g(v_h)+g(v_h)-g(v)\rangle$$

$$+\langle Au_h, g(u_h)-g(v)+g(v)-g(v_h)\rangle$$

$$= \langle Tu-Tu_h, g(v)-g(u_h)\rangle + \langle Tu-Tu_h, g(u)-g(v_h)\rangle$$

$$+\{\langle Au, g(u)-g(v_h)\rangle + \langle Au, g(v_h)-g(v)\rangle\}$$

$$+\{\langle Au_h, g(u_h)-g(v)\rangle + \langle Au_h, g(v)-g(v_h)\rangle\}$$

$$- \langle Tu, g(u)-g(v_h)\rangle - \langle Tu_h, g(u_h)-g(v)\rangle$$

$$= \langle Tu-Tu_h, g(v)-g(u_h)\rangle + \langle Tu-Tu_h, g(u)-g(v_h)\rangle$$

$$+\langle Au-Tu, g(u)-g(v_h)\rangle + \langle Au_h-Tu_h, g(u_h)-g(v)\rangle$$

$$+\langle Au-Au_h, g(v_h)-g(v)\rangle$$

or l.h.s of (2.3.5)

$$\leq \beta\gamma||u-u_h||\ ||v-u_h|| + \beta\gamma||u-u_h||\ ||u-v_h|| +$$

$$+\gamma||Au-Tu||\ ||u-v_h|| + \gamma||Au_h-Tu_h||\ ||u_h-v||$$

$$+\sigma\gamma||u-u_h||\ ||v_h-v|| \tag{2.3.6}$$

and r.h.s of (2.3.5)

$$= \langle Tu-Tu_h,\ g(u)-g(u_h)\rangle$$

$$= \langle Tu-Tu_h,\ u-u_h\rangle - \langle Tu-Tu_h,\ (u-u_h)-(g(u)-g(u_h))\rangle$$

$$\geq \alpha||u-u_h||^2 - ||Tu-Tu_h||\ ||(u-u_h)-(g(u)-g(u_h))||$$

$$\geq \alpha||u-u_h||^2 - \beta\sqrt{1-2\delta+\gamma^2}\ ||u-u_h||^2$$

or r.h.s $\geq (\alpha-\beta\sqrt{1-2\delta+\gamma^2})\ ||u-u_h||^2$ (2.3.7)

The result follows from (2.3.6) and (2.3.7) and keeping in mind the Young inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, for points a,b and $\varepsilon > 0$

(2.3.4) is a special case of (2.3.3) for $v = v_h$ if we keep in mind that $||v_h-u_h||_H \leq ||v_h-u||_H + ||u-u_h||_H$.

REMARK 2.3.1. If $g = I$ $Au = f$ and $T = A$ then theorem (2.3.1) reduces to theorem (2.2.1).

2.4. AN APPLICATION

We consider the special case when the problem (2.2.1) is defined in the following setting. Let $H = H_o^1(\Omega)$, $H' = H^{-1}(\Omega)$ and $W = W' = L^2(\Omega)$. For $u,v \in H_o^1(\Omega)$, let

$$a(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}(x)\, u_{x_i} v_{x_j}\, dx + \int_\Omega C(x)\, uv dx$$

where $\sum_{i,j=1}^{2} a_{ij}\, \xi_i \xi_j \geq \bar{\alpha} |\xi|^2$, $\forall\ \xi \in R^2$ and some $\alpha > 0$

suppose that $a_{ij} = a_{ji}$ so that $a(u,v)$ is symmetric and let $a_{ij} \in C^1(\bar{\Omega})$ and $C \in L^\infty(\Omega)$. Then there exist a constant $\beta > 0$ such that

$$|a(u,v)| \leq \beta ||u||_{1,\Omega} ||v||_{1,\Omega} \quad \forall\ u,v \in H_0^1(\Omega)$$

Further, suppose that $C(x) \geq \lambda$ where λ is either negative or nonnegative with sufficiently small absolute value so that $a(u,v)$ is coercive on $H_0^1(\Omega)$, i.e.

$$a(u,v) \geq \alpha ||v||^2 \quad \forall\ v \in H_0^1(\Omega) \text{ and } \alpha > 0$$

is a constant

Let A be an operator defined by $Au = -\sum_{ij=1}^{2} (a_{ij} x_i) x_j + cu$ Then, for $u,v \in H_0^1(\Omega)$, $a(u,v) = (Au,v)$, and since $a_{ij} = a_{ji}$, A is a self adjoint second order elliptic operator mapping $H_0^1(\Omega) \to H^{-1}(\Omega)$.

Finally, suppose that $f \in L^2(\Omega)$ and χ is a given function in $H^2(\Omega)$ such that $\chi \leq 0$ on $\partial(\Omega)$. In terms of the above notation, we will consider the approximation of problem (2.2.1) where,

$$K = \{v \in H_o^1(\Omega) : v \geq \chi \quad \text{a.e. in } \Omega\}$$

The approximation scheme we use for this problem will be of general type described as in Section 2. i.e., we will replace problem (2.2.1) by minimization problem (2.2.2) which can be solved by mathematical techniques. Here we need only explicit choices for finite dimensional subspace $V_h \subset H_o^1(\Omega)$ and approximate convex set $K_h \subset V_h$. We first describe the construction of V_h.

Let h, $0 < h < 1$, be a parameter and, for each value of h let Ω_h be a polygon inscribed in Ω with all its vertices lying on $\partial\Omega$ and each side of the polygon of length less or equal to h. Now subdivide Ω_h in to triangles T_q $q = 1,2,\ldots N_h$ such that the following two conditions are satisfied.

(i) The ratio of any two triangles sides in the triangulation is bounded by a const b independent of h.

(ii) All the angles in the triangulation are greater or equal to some angle Φ independent of h.

Using the terminology [60], we will henceforth refer to a triangulation satisfying condition (i) and (ii) as Φ-b regular.

Now, define $V_h = \{v_h : v_h$ is linear in each triangle T_q, $q = 1,2, \ldots N_h$, continuous on Ω and $v_h = 0$ in $\Omega - \Omega_h\}$. Clearly, V_h is a finite dimensional subspace of $H_o^1(\Omega)$.

Finally, define $K_h = \{v_h \in V_h : v_h \geq \chi$ at every vertex of each triangle T_q, $q=1,2,\ldots N_h\}$. Clearly K_h is closed convex subset of V_h and satisfies condition (i) of Section 2. Since K_h reduces to a finite number of linear constraints the approximation minimization problem (2.2.2) becomes a problem in quadratic programming.

Now we will state the following error estimate for the approximation scheme defined as above.

THEOREM 2.4.1. [24]. Let u and u_h be the solution of problem (2.2.1) and (2.2.2) respectively with K and K_h defined as above. There exist a constant C independent of u and depending upon the data Ω, f, χ, a_{ij} and c such that

$$||u-u_h||_{1,\Omega} \leq Ch.$$

CHAPTER - III

WIENER-HOPF EQUATION AND VARIATIONAL INEQUALITIES

3.1. INTRODUCTION.

Variational inequalities and Wiener-Hopf equations are of course two different subjects. In this chapter we discuss the equivalence between the variational inequality with Wiener-Hopf equation in the sense that, if one of them has a solution so does the otherone This allows one's to study these subjects with a unified point of view and to transplant techniques suitable for equations into inequality. Shi [73], has shown that the variational inequality (3.2.1) are equivalent to Wiener-Hopf equation and as an application he discussed the convergence of an iteration scheme for parabolic variational inequalities with unilateral constraints.

Recently, Noor [67], has shown, that general variational inequalities are equivalent to general Wiener-Hopf equations. This equivalence is useful for the numerical point of views and he suggests and analyse a number of new iterative algorithms for computing approximate solutions of general variational inequalities. He also study the conditions under which the approximate solutions obtained from iterative algorithms converges to the exact solution of the general variational inequalities.

In Section 3.2, we discuss the theory of equivalence of variational inequality with Wiener-Hopf equations. The equivalence of general variational inequality with general Wiener-Hopf equation is discussed in Section 3.3. In Section 3.4 of this chapter convergence analysis is discussed.

3.2. EQUIVALENCE OF VARIATIONAL INEQUALITY WITH WIENER-HOPF EQUATION

Let P_k be the projection operator and $Q_k = I-P_k$, where I is the Identity mapping. Let A be an arbitrary operator defined on $D(A) \subset H$, which maps $D(A)$ in to H. For a given $f \varepsilon H$, consider the variational inequality.

Find $u \varepsilon D(A) \cap K$ such that

$$(Au,\ v-u) \geq (f,\ v-u),\ \forall\ v \varepsilon K \tag{3.2.1}$$

The nonlinear equation,

$$(AP_k + Q_k)\ v = f \tag{3.2.2}$$

is called a generalized Wiener-Hopf equation. Here $v \varepsilon H$, is to be found. If K is a closed subspace and the projection operators P_k and Q_k are linear operators, then (3.2.2) reduces a linear equation, provided A is linear. The different types of examples of generalized linear Wiener-Hopf equations are given in [92].

Here we present some results showing equivalence between variational inequality problem and Wiener-Hopf equation.

THEOREM 3.2.1. [73]. The variational inequality (3.2.1) has a solution u if and only if the Wiener-Hopf equation (3.2.1) has a solution v. Moreover, $v = u+f-Au$, and $u = P_k v$

THEOREM 3.2.2 [73]. The variational inequality (3.2.1) has a unique solution for each f if and only if the Wiener-Hopf equation (3.2.2) has a unique solution for each f. In this case, the operator AP_k+Q_k is surjective and one to one. The solution of (3.2.1) is given by the farmula

$$u = P_k(AP_k+Q_k)^{-1}f \tag{3.2.3}$$

THEOREM 3.2.3 [73]. Suppose A is linear, one-to-one mapping from H on to H and K is closed convex cone with vertex at the origin. Then a necessary condition for (3.2.1) to have a solution for each f is that the operator A admits a Wiener-Hopf factorization; i.e.

$$A = A_- A_+ \quad \text{with} \quad A_+ P_k = P_k \;\; A_- Q_k = Q_k \tag{3.2.4}$$

3.3. EQUIVALENCE OF GENERAL VARIATIONAL INEQUALITY WITH WIENER-HOPF EQUATION

Let $T, g : H \to H$ are the nonlinear continuous operators, we consider the problem to find $u \in H$ such that $g(u) \in K$, and

$$\langle T(u),\, g(v) - g(u) \rangle \geq 0, \text{ for all } g(v) \in K \tag{3.3.1}$$

The inequality (3.3.1) is known as general variational inequality, which was studied by Noor [64], in the study of odd order obstacle problems.

Let P_k be the projection operator of H in to K, and let $Q_k = I - P_k$, where I is the identity operator. $T, g : H \to H$, are nonlinear operators. If g^{-1} exists, then we consider the problem to

Find $z \in H$ such that

$$T\, g^{-1} P_k z + \rho^{-1} Q_k z = 0 \tag{3.3.2}$$

where $\rho > 0$, is a constant.

Equation of the type (3.3.2) are called general Wiener-Hopf equations. If $g \equiv I$, the identity operator, then the problem (3.3.2) is equivalent to finding $z \in H$ such that

$$TP_k z + \rho^{-1} Q_k z = 0 \qquad (3.3.3)$$

The detail general theory and applications of Wiener-Hopf equation can be found in [92].

Here we state an equivalence result.

THEOREM 3.3.1 [67]. The general variational inequality (3.3.1) has a solution $u \in H$ such that $g(u) \in K$, if and only if the Wiener-Hopf equation (3.3.2) has a solution $z \in H$, where

$$z = g(u) - \rho T(u) \qquad (3.3.4)$$

$$g(u) = P_k z \qquad (3.3.5)$$

where P_k is projection of H in to K and $\rho > 0$ is a constant.

PROOF. Let $u \in H$ be such that $g(u) \in K$ is a solution of (3.3.1). Then by Lemma (1.2.1), it follows that

$$g(u) = P_k[g(u) - \rho Tu] \qquad (3.3.6)$$

Using $Q_k = I - P_k$ and applying (3.3.6) repeatedly, we obtain

$$Q_k[g(u) - \rho Tu] = g(u) - \rho Tu - P_k[g(u)-\rho Tu]$$
$$= -\rho Tu$$
$$= -\rho T\ g^{-1}P_k[g(u)-\rho Tu]$$

from which it follows that

$$Tg^{-1}\ P_k z + \rho^{-1}\ Q_k z = 0$$

where $z = g(u) - \rho Tu$,

Conversely, let $z \in H$ be a solution of (3.3.2). Then we have

$$\rho\ T\ g^{-1}P_k z = -Q_k z = P_k z - z \tag{3.3.7}$$

Now from (3.3.7) and Lemma (1.2.1), for all $g(v) \in K$, we obtain

$$0 \leq \langle P_k z-z,\ g(v)-P_k z\rangle = \langle \rho Tg^{-1}P_k z,\ g(v)-P_k z\rangle$$

from which it follows that

$$\langle Tg^{-1}P_k z,\ g(v)-P_k z\rangle \geq 0, \quad \text{for all} \quad g(v) \in K$$

Thus, $g(u) = P_k z$ is a solution of (3.3.1), and from (3.3.7), we have

$$z = g(u) - \rho Tu$$

REMARK 3.3.1. If $g = I$, $T = A$, then theorem (3.3.1) reduces to theorem (3.2.1) of Shi [73]

3.4. CONVERGENCE ANALYSIS

In this section, by using the equivalence and by an appropriate rearrangement, Noor [67], suggests a number of iterative algorithms for solving variational inequalities (3.3.1). We also state a result showing that the approximate solutions obtained from iterative algorithms converges to the exact solution of the general Wiener-Hopf equation (3.3.2).

(i) The Wiener-Hopf equation (3.3.2) can be written as

$$Q_k z = -\rho T g^{-1} P_k z$$

which implies that, using (3.3.5),

$$\begin{aligned} z &= P_k z - \rho T\, g^{-1}\, P_k z \\ &= g(u) - \rho T(u) \end{aligned} \tag{3.4.1}$$

Using this farmulation Noor [67], suggests the following iterative algorithm for solving the general variational inequalities (3.3.1).

ALGORITHM 3.3.1. For a given $z_o \in H$, compute z_{n+1} by the iterative scheme

$$g(u_n) = P_k z_n \tag{3.4.2}$$

$$z_{n+1} = g(u_n) - \rho T(u_n), \quad n = 0,1,2,\ldots. \tag{3.4.3}$$

(ii) By an appropriate rearrangement, the general Wiener-Hopf equation (3.3.1) can be written in the following form:

$$z = P_k z - \rho T g^{-1} P_k z + (I-\rho^{-1}) Q_k z$$

$$= u - \rho Tu + (I-\rho^{-1}) Q_k z \tag{3.4.4}$$

By using (3.3.5) in this formulation, Noor [67] suggests and propose the following iterative algorithm

ALGORITHM 3.4.2. For a given $z_o \in H$, compute z_{n+1} by the iterative schemes

$$g(u_n) = P_k z_n$$

$$z_{n+1} = u_n - \rho T u_n + (I-\rho^{-1}) Q_k u_n, \quad n = 0,1,2,\ldots \tag{3.4.5}$$

(iii) If the operator T is linear and T^{-1} exists, then the general Wiener-Hopf equations can be written as

$$z = (I-\rho^{-1} g\, T^{-1}) Q_k z \tag{3.4.6}$$

By using this formulation Noor [67] suggests the following iterative algorithms for solving general variational inequalities.

ALGORITHM 3.4.3. For a given $z_o \in H$, compute z_{n+1} by the following iterative scheme.

$$z_{n+1} = (I-\rho^{-1}gT^{-1})\, Q_k z_n, \quad n = 0,1,2,\ldots. \qquad (3.4.7)$$

Now here we present a convergence result.

THEOREM 3.4.1 [67]. Let $T,g : H \to H$ be both strongly monotone and Lipschitz continuous operators. If z_{n+1} is obtained from the iterative scheme (3.4.2)-(3.4.3), and if $z \in H$ is the exact solution of the general Wiener-Hopf equation (3.3.1), then

$$z_{n+1} \to z, \text{ strongly in } H,$$

for

$$|\rho-\alpha/\beta^2| < \sqrt{\alpha^2-4\beta^2(k-k^2)}/\beta^2 \;,\; \alpha > 2\beta\sqrt{k-k^2},\; k > 1$$

PROOF. Let $z \in H$ satisfy the general Wiener-Hopf equation (3.3.2)., Since equation (3.3.2) can be written as (3.3.5) and (3.4.1). Hence from (3.4.1) and (3.4.3), we have

$$||z_{n+1}-z|| = ||g(u_n)-g(u)-\rho(Tu_n-Tu)||$$

$$\leq ||u_n-u-(g(u_n)-g(u))||$$

$$+||u_n-u-\rho(Tu_n-Tu)|| \qquad (3.4.8)$$

Since T and g are strongly monotone and Lipschitz continuous, that is, there exist constant $\alpha > 0$, $\beta > 0$ such that, for all $u,v \in H$

$$<Tu-Tv,\ u-v> \geq \alpha||u-v||^2$$

$$||Tu-Tv|| \leq ||u-v||$$

so

$$||u_n-u-\rho(Tu_n-Tu)||^2 \leq ||u_n-u||^2 - 2\rho <Tu_n-Tu,u_n-u> + ||Tu_n-Tu||^2$$

$$\leq (1-2\rho\alpha+\beta^2\rho^2)\ ||u_n-u||^2 \qquad (3.4.9)$$

and similarly

$$||u_n-u-(g(u_n)-g(u))||^2 \leq (1-2\sigma+\delta^2)||u_n-u||^2 \qquad (3.4.10)$$

where σ and δ are the strong monoticity constant and Lipschitz continuity constant of operator g.

Combining (3.4.8), (3.4.9), (3.4.10), we obtain

$$||z_{n+1}-z|| \leq \{k+t(\rho)\}||u_n-u|| \tag{3.4.11}$$

where $K = \sqrt{1-2\sigma+\delta^2} \neq 1$, $t(\rho) = \sqrt{1-2\rho\alpha+\rho^2\beta^2}$

From (3.4.2) we have,

$$||u_n-u|| \leq ||u_n-u-(g(u_n)-g(u))||+||P_k z_n-P_k z||$$

$$\leq K||u_n-u|| + ||z_n-z||$$

Using (3.4.10), from which it follows that

$$||u_n-u|| \leq [\frac{1}{(1-k)}] ||z_n-z|| \tag{3.4.12}$$

Thus from (3.4.11) and (3.4.12), we obtain

$$||z_{n+1}-z|| \leq \{(k+t(\rho))/_{(1-k)}\} ||z_n-z|| = \theta||z_n-z||$$

where $\theta = (k+t(\rho))/_{(1-k)} < 1$

for $|\rho-{}^{\alpha}/_{\beta^2}| < \sqrt{(\alpha^2-4\beta^2(k-k^2))}|\beta^2$, $\alpha > 2\beta\sqrt{k-k^2}$, $k < 1$

Since $\theta < 1$, so mapping defined by (3.4.1) has a fixed point z which is the solution of (3.3.2). Furthermore, it also follows that $z_{n+1} \rightarrow z$ strongly in H.

CHAPTER-IV

VECTOR VARIATIONAL INEQUALITIES

4.1. INTRODUCTION

In 1980, F. Gianessi [30] has introduced the vector variational inequality in a finite dimensional Euclidean space with further application. From that time on, in a general setting Chen and Cheng [13]. Chen and Yang [15] and Chen [16] have derived equivalence between the vector variational inequality and the vector extremum problem, the equivalence between the vector variational inequality and the vector complimentarity problem and, proved the existence of the solution of vector variational inequality. Recently Siddiqi, Ansari and Khaliq [87] introduced more general form of vector variational inequality corresponding to general variational inequality introduced by Noor [64], which includes the vector variational inequality studied by Chen [16] as a special case and they proved the existence of its solution which may be seen an extension of the Isac's theorems [40] on the existence of solution for a general variational inequality problem.

In Section 4.2, of this chapter we study the existence of solution for vector variational inequality. In Section 4.3, we discuss the existence of solution for general vector variational inequality. In the last section, we study the existence without convexity.

4.2. EXISTENCE THEORY FOR SOLUTION OF VECTOR VARIATIONAL INEQUALITY

Let X and Y be two ordered Banach spaces. Let $K \subseteq X$ be nonempty, closed and convex, let $T:K \to L(X,Y)$ be a mapping where $L(X,Y)$ is the space of all linear continuous operators from X into Y. Let $\{C(u): u \in K\}$ be a family of closed, pointed convex cones of Y such that $\text{int}\, C(u) \neq \phi$, $\forall\, u \in K$, where int denotes interior.

Consider the vector variational inequality problem:

$$(\text{VVI}) \quad u_0 \in K, \ \langle T(u_0), u-u_0 \rangle \notin -\text{int}\, C(u_0) \quad \forall\, u \in K \qquad (4.2.1)$$

where $\langle T(u),v \rangle$ denotes the evaluation of linear operator $T(u)$ at v. Hence, $\langle T(u),. \rangle \in Y$. The above problem is generalization of the classical variational inequality

$$u \in K, \ \langle T(u), v-u \rangle \geq 0, \quad \forall\, v \in K \qquad (4.2.2)$$

when $Y = R$, $X = R^n$, $L(X,Y) = X^*$, $C(u) = R_+$.

DEFINITION 4.2.1 [16]. Let $T:X \to L(X,Y)$. Let $C_- = \bigcap_{u \in X} C(u)$ be nonempty. T is said to be C_--monotone if and only if

$$\langle T(v)-T(u),\ v-u\rangle \in C_- \quad \forall\ u,v \in K$$

The following lemma plays an important role to prove the theorems (4.2.1) and (4.3.2).

LEMMA 4.2.1 [16]. Let (V,P) be ordered topological vector space with a closed, pointed and convex cone with int $P \neq \phi$. Then $u,v \in V$, we have

(i) $u-v \in \text{int } P$ and $u \notin \text{int } P \implies v \notin \text{int } P$;

(ii) $u-v \in P$ and $u \notin \text{int } P \implies v \notin \text{int } P$;

(iii) $u-v \in -\text{int } P$ and $u \notin -\text{int } P \implies v \notin -\text{int } P$;

(iv) $u-v \in -P$ and $u \notin -\text{int } P \implies v \notin -\text{int } P$.

Now we state the generalization of Minty Lemma [51] for vector variational inequality which plays an important role in the theory of variational inequality.

LEMMA 4.2.2 [16]. Let $T:X \to L(X,Y)$ be a C_--monotone and hemi-continuous mapping on X. Then the following problems (I) and and (II) are equivalent for any convex subset K in X:

$$u \in K,\ \langle T(u),v-u\rangle \notin -\text{int } C(u),\ \forall\ v \in K \qquad \text{(I)}$$

$$u \in K,\ \langle T(v),v-u\rangle \notin -\text{int } C(u),\ \forall\ v \in K \qquad \text{(II)}$$

The following existence theorem is proved by Chen [16].

THEOREM 4.2.1 [16]. Let X be a reflexive Banach space. Let $K \subset X$ be a nonempty, bounded, closed, convex subset in X. Let $C:K \to 2^Y$ be multivalued function such that, for every $u \in K$, $C(u)$ is closed, pointed and, convex cone with $\operatorname{int} C \neq \phi$. Let the multivalued function $W(u) = Y \setminus \{-\operatorname{int} C(u)\}$ be upper semi-continuous on K. Then, the vector variational inequality (4.2.1) is solvable.

DEFINITION 4.2.3 [16]. Let K be a convex and unbounded subset X. We shall say that $T:K \to L(X,Y)$ is coercive on K iff there exist $u_o \in K$ and $s \in C_+^* \setminus \{0\}$, such that

$$\langle soT(u)-soT(u_o),\ u-u_o\rangle / ||u-u_o|| \to \infty$$

whenever $u \in K$ and $||u|| \to \infty$, here

$$soT(u) = s(T(u))$$

and

$$C_+^* = \{\lambda \in Y^*:\langle\lambda,v\rangle \geq 0, \quad \forall\ v \in C_+\}$$

Now we study the existence of solution of vector variational inequality in infinite dimensional spaces by using theorems of alternative.

Let X and Y be topological vector spaces, K a non-empty subset of X, H a nonempty subset of Y, and $M:K \to Y$ is a mapping. We study the condition under which the system
(S) $\exists\ v \in K$, such that $M(v) \in H$, will have or will not have solutions.

DEFINITION 4.2.4 [16]. Let Z be a subset of R. The real function $w:Y \to R$ is called a weak separation function if and only if

$$H^w = \{h \in Y : w(h) \notin Z\} \supseteq H$$

The real function $s:Y \to R$ is called a strong separation function if and only if,

$$H^s = \{h \in Y : s(h) \notin Z\} \subseteq H$$

LEMMA 4.2.3 [16]. Let the set H, K, Z and M be given then,

(i) The system (S) and the system

$$w(M(v)) \subseteq Z, \quad \forall\ v \in K$$

are not simultaneously possible, whatever the weak separation function w may be.

(ii) The system (S) and the system

$$s(M(v)) \subseteq Z \quad \forall\ v \in K,$$

are not simultaneously impossible, whatever the strong separation s may be

THEOREM 4.2.2 [16]. Let X be a reflexive Banach space, Y a Banach space. Let $K \subset X$ be closed, convex and unbounded sub set. Let $C:K \to 2^Y$ be a multivalued function such that, for every $u \in K$, $C(u)$ is a closed, pointed, and convex cone with int $C(u) \neq \Phi$, and let $C_- = \bigcap_{u\in K} C(u)$ be such that int $C_- = \Phi$. Let $C_+ = \text{conv}\,\{C(u): u \in K\}$ be a closed, pointed and convex cone in Y. Let $T:K \to L(X,Y)$ be C_--monotone, hemicontinuous and coercive on K. Then vector variational inequality (4.2.1) is solvable.

REMARK 4.2.1. If $Y = R$, $C(u) = R_+$, $\forall\, u \in K$, the vector variational inequality (4.2.1) reduces to usual variational inequality and theorem (4.2.1) reduces to the following theorem.

THEOREM 4.2.3 [16]. Let X be a reflexive Banach space. Let K be a nonempty, bounded and closed convex subset in X; and let $T:K \to X^*$ be monotone and hemicontinuous on K. Then the variational inequality

$$u_o \in K, \ \langle T(u_o),\ u-u_o\rangle \geq 0, \quad \forall\, u \in K$$

is solvable.

Obviously, the hemicontinuity of T in theorem (4.2.1) is equivalent to the continuity for each one dimensional flat $L \subset X$. Thus the theorem (4.2.3) is essentially the Hartman Stampacchia theorem [36].

4.3. EXISTENCE OF SOLUTION FOR GENERAL VECTOR VARIATIONAL INEQUALITY

Let X be Hausdorff topological vector space and Y be an ordered Hausdorff topological vector space. Let K be a nonempty closed convex subset of X, $T:K \to L(X,Y)$ is a mapping Let $\{C(u):u \in K\}$ be a family of closed, pointed convex cone in Y, with $\operatorname{int} C(u) \neq \phi$ for every $u \in K$. The bilinear form $\langle .,. \rangle$ is supposed to be continuous.

Consider the general vector variational inequality problem:

$$\text{(GVVIP)} \quad \begin{cases} \text{Find } u_o \in K \text{ such that} \\ \langle T(u_o), u-g(u_o) \rangle \not\in -\operatorname{int} C(u_o) \\ \qquad\qquad \text{for all } u \in K \end{cases} \tag{4.3.1}$$

where $g:K \to K$ is a mapping, $\langle T(u), . \rangle \in Y$.

SPECIAL CASES:

(i) For $g(u) = u \in K$, the (GVVIP) (4.3.1) is equivalent to (4.2.1).

(ii) If $Y = R$, $C(u) = R_+$, for all $u \in K$, then (GVVIP) (4.3.1) reduces to finding $u_o \in K$ such that

$$\langle T(u_o), u-g(u_o)\rangle \geq 0, \text{ for all } u \in K \qquad (4.3.2)$$

The inequality (4.3.2) is known as general variational inequality introduced and studied, separately, by Isac [40] and Noor [64].

(iii) If $Y = R$, $C(u) = R_+$, for all $u \in K$ and g is the identity mapping then general vector variational inequality problem (4.3.1) become usual variational inequality (4.2.2) introduced and studied by Hartman and Stampacchia [36].

Now we have the following existence results:

THEOREM 4.3.1 [87]. If $T:K \to L(X,Y)$ and $g:K \to K$ are continuous mappings, the multivalued map $W(u) = Y\setminus\{-\text{int } C(u)\}$ is upper semicontinuous on K and $\langle T(u), u-g(u)\rangle \notin -\text{int } C(u)$ for every $u \in D$. Then there exists an $u_o \in D \subset K$, such that

$$\langle T(u_o), u-g(u_o)\rangle \notin -\text{int } C(u_o), \text{ for all } u \in D \subset K$$

THEOREM 4.3.2 [87]. Assume that

1° The mapping $g:K \to K$ and $T:K \to L(X,Y)$ are continuous.

2° $C:K \to Y$ is multivalued mapping such that for every $u \in K$, $C(u)$ is closed pointed convex cone with int $C(u) \neq \phi$.

3° $W:K \to Y$ is an upper semicontinuous multivalued mapping defined as $W(u) = Y \setminus \{-\text{int } C(u)\}$;

4° There exists a function $h:X \times X \to Y$ such that

(i) $\langle T(u), v-g(u)\rangle - h(u,v) \in -\text{int } C(u)$,

(ii) the set $\{v \in K : h(u,v) \in -\text{int } C(u)\}$, is convex for every $u \in K$,

(iii) $h(u,u) \notin -\text{int } C(u)$, for all $u \in K$,

(iv) there exists a nonempty compact convex subset $D \subset K$ such that for every $u \in K \setminus D$, there exist a point $v \in D$ such that

$$\langle T(u), v-g(u)\rangle \in -\text{int } C(u)$$

Then there exist a point $u_o \in D$ such that

$$\langle T(u_o), v-g(u_o)\rangle \notin -\text{int } C(u_o), \text{ for all } v \in K$$

COROLLARY 4.3.1. Suppose that condition (1°)-(3°) of theorem (4.3.2) is satisfied and also assume

4°. $\langle T(u), u-g(u)\rangle \notin -\text{int } C(u), \forall u \in K$

5°. There exist a nonempty compact convex subset $D \subset K$ such that for every $u \in K \setminus D$, there exist a point $v \in D$ with

$$\langle T(u), v-g(u)\rangle \in -\text{int } C(u)$$

Then there exist a point $u_o \in D \subset K$ such that

$$\langle T(u_o), v-g(u_o)\rangle \notin -\text{int } C(u_o) \quad \text{for all } v \in K$$

4.4. EXISTENCE THEORY WITHOUT CONVEXITY.

In this section, we study the existence theorems for vector variational and general vector variational inequality by replacing convexity assumption with merely topological properties.

Now we consider the following vector variational inequality (VVI)'

$$u_o \in X, \ \langle T(u_o), u-u_o\rangle \notin -\text{int } P, \ \forall u \in X$$

where (Y,P) is an ordered Banach space, with int $P \neq \phi$; and $T: X \to L(X,Y)$ is a mapping.

Now we state the following existence results.

THEOREM 4.4.1 [87]. Let $(X,\{\Gamma_A\})$ be an H-space, and (Y,P) be an ordered Banach space with a closed, pointed, and convex cone P such that int $P \neq \phi$, let $T:X \to L(X,Y)$ be a continuous mapping such that

(i) $\forall\ v \in X$, $B_v = \{u \in X,\ \langle T(v),\ u-v\rangle \in \text{int } P\}$
is H-convex or empty

(ii) there exist a compact set $L \subset X$ and an H-compact set $K \subset X$, such that, for every weakly H-convex set D with $K \subset D \subset X$, we have

$\{v \in D : \langle T(v), u-v\rangle \notin -\text{int } P,\ \forall\ u \in X\} \subset L$

Then, the vector variational inequality (VVIP)' (4.4.1) is solvable.

We now consider the general vector variational inequality

(GVVI)' $\quad u_o \in X,\ \langle T(u_o),\ u-g(u_o)\rangle \notin -\text{int } P,\ \forall\ u \in X,$

where (Y,P) is an ordered Banach space with int $P \neq \phi$, and $T:X \to L(X,Y)$ and $g:X \to X$ are the mappings.

Then we have the following existence result.

THEOREM 4.4.2 [87]. Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space with a closed pointed convex cone and a nonempty interior int $P \neq \phi$. Assume that

1. The mappings $T: X \to L(X,Y)$ and $g: X \to Y$ are continuous.
2. $\langle T(v), v-g(v)\rangle \notin -\text{int } P$ for all $v \in X$;
3. for every $v \in X$, $B_v = \{u \in X, \langle T(v), u-g(v)\rangle \in \text{int } P\}$ is H-convex or empty.
4. There exist a compact set $L \subset X$ and an H-compact set $K \subset X$ such that for every weakly H-compact set D with $K \subset D \subset X$.

   $\{v \in D : \langle T(v), u-g(v)\rangle \notin -\text{int } P, \text{ for all } u \in D\} \subset L$

Then general vector variational inequality (GVVI)', (4.4.2) is solvable.

CHAPTER - V

VARIATIONAL-LIKE INEQUALITIES

5.1. INTRODUCTION

In recent years the theory of variational inequalities has been extended and generalized in various directions, because of its application in different branches of science, engineering, optimization, economics, equilibrium theory etc. The variational-like inequality problem is one of the generalized form of the variational inequality problem, which was introduced and studied by Parida, Sahoo and Kumar [69] in 1989. They proved the existence of its solution in R^n and have shown its relationship with mathematical programming. Recently, Siddiqi, Ansari and Khaliq [85] developed a theory for existence the solution of variational-like inequalities in reflexive Banach spaces and topological vector spaces. Yang and Chen [93] introduced a wider class of nonconvex functions known as semi-preinvex function which includes the pre-invex function [17], and arcwise connected convex functions [5]. They have shown that variational-like inequality is a necessary condition and also a sufficient condition for a mathematical programming for invex function [52]. They also proved some existence results in R^n.

In Section 5.2, we study the solution of variational-like inequality in R^n. In Section 5.3, we study the existence of a solution of variational-like inequality problem in the setting of reflexive Banach spaces. Existence theory in Hausdorff topological vector space is discussed in Section 5.4. While Section 5.5 deals with existence theory without convexity.

5.2. EXISTENCE THEORY OF VARIATIONAL-LIKE INEQUALITY IN R^n

Let K be closed convex subset in R^n and $T:K \to R^n$ and $\eta:K \times K \to R^n$ be continuous maps then the problem of finding $u_o \in K$, such that

$$<T(u_o),\eta(u,u_o)> \geq 0, \forall u \in K \tag{5.2.1}$$

is called variational-like inequality problem studied by Parida, Sahoo and Kumar [69].

We need the following definitions:

DEFINITION 5.2.1 [69]. A mapping $T:K \to R^n$ is said to be η-monotone if there exists a continuous map $\eta:K \times K \to R^n$ such that

$$\langle T(v),\eta(u,v)\rangle + \langle T(u),\eta(v,u)\rangle \leq 0 \qquad (5.2.2)$$

T is said to be strictly η-monotone over K if the equality hold in (5.2.2) only when $v = u$.

NOTE: If $\eta(u,v) = u-v$ definition (5.2.1) reduces to definition of monotone functions.

DEFINITION 5.2.2 [69]. Let $\Psi: K \to R^n$ be differentiable. Then Ψ is η-convex on K if there exist a continuous map $\eta : K \times K \to R^n$ such that

$$\Psi(v)-\Psi(u) \geq \langle \nabla\Psi(u),\eta(v,u)\rangle, \quad \forall\ u,v \in K \qquad (5.2.3)$$

It is known that Ψ is convex on K if $\nabla\Psi$ is monotone on K.

Now we state the following existence theory.

THEOREM 5.2.1.[69]. Let K be a compact and convex set in R^n, and let $T: K \to R^n$ and $\eta: K \times K \to R^n$ be two continuous maps. Suppose that

$$\langle T(u),\eta(u,u)\rangle = 0, \quad \forall\ u \in K \qquad (5.2.4)$$

for each fixed $u \in K$, the function

$\langle T(u), (v,u)\rangle$ is quasiconvex in $v \in K$.

Then $\exists$ $u_o \in K$ such that

$$\langle T(u_o),\eta(u,u_o)\rangle \geq 0 \quad \forall\ u \in K$$

CONDITION 5.2.1. Let K be closed convex set in R^n. Let $T:K \to R^n$ and $\eta:K \times K \to R^n$ be two continuous maps, such that

(I) $\langle T(u),\eta(u,u)\rangle = 0$, and

(II) for each fixed $u \in K$, the function $\langle T(u),\eta(v,u)\rangle$ is convex in $v \in K$

Notice that $K_r = \{u:u \in K$ and $||u|| < r$, for $r > 0\} \subseteq K$ is compact and convex, and hence, by theorem (5.2.1) there exists at least one $u_r \in K_r$ such that

$$\langle T(u_r),\eta(u,u_r)\rangle \geq 0, \quad \forall\, u \in K_r \tag{5.2.5}$$

Whenever condition (5.2.1) is satisfied.

Since compactness is very strong condition as in theorem (5.2.1), now by relaxing the compactness condition, we state the following theorem (5.2.2) which is generalization of (5.2.1).

THEOREM 5.2.2 [69]. Let K,T and η be such that condition (5.2.1) is satisfied. A necessary and sufficient condition that there exist a solution to (5.2.1) is that there exist an $r > 0$ such that a solution $u_r \in K_r$ of (5.2.5) satisfies the estimate

$$||u_r|| < r$$

Now, by using theorem (5.2.2) we give three other important sufficient condition for the existence of a solution to (5.2.1)

THEOREM 5.2.3 [69]. Let K, η and T be such that the condition (5.2.1) is satisfied. Then the variational-like inequality problem has a solution under each of the following conditions.

(I) there exist a $x \in K$ and a scalar $r > ||x||$ such that

$$\langle T(u), \eta(x,u)\rangle \leq 0, \text{ for all } u \text{ with } ||u|| = r \qquad (5.2.6)$$

(II) for some constant $r > 0$, and for each $u \in K$ with $||u|| = r$, there is a $x \in K$ with $||u|| < r$ and $\langle T(u), \eta(x,u)\rangle \leq 0$

(III) there exist a nonempty, compact and convex subset C of K with the property that for every $u \in K \setminus C$ there xists a $x \in C$ such that

$$\langle T(u), \eta(x,u)\rangle < 0 \qquad (5.2.7)$$

Generally, the solution of the variational-like inequality may not be unique. There is however a very natural condition which ensures uniqueness.

THEOREM 5.2.4.[69]. Let $T:K \to R^n$ and $\eta:K \times K \to R^n$ be continuous over the closed convex set K. If T is strictly monotone over K, then there can exist at most one solution to the variational like inequality problem.

A well known fact in mathematical programming is that the variational inequality problem has a close relation with the optimization problem.

DEFINITION 5.2.3 [93]. Let $f(u)$ on R^n is real differentiable function with the gradient $\nabla f(u)$, which satisfies: for every $u,v \in R^n$, there exist a vector $\eta(v,u) \in R^n$, such that

$$f(v) \geq f(u) + \langle \nabla f(u), \eta(v,u) \rangle$$

Then the function f is called invex function

DEFINITION 5.2.4 [93]. A real differentiable function f defined on R^n, $f:K \subseteq R^n \to R$ is invex if, for $u,v \in K$, there exist a vector $\eta(v,u) \in R^n$, such that, $\forall\ \alpha \in [0,1]$, $u+\alpha\eta(v,u) \in K$, and

$$f(u+\alpha\eta(v,u)) \leq \alpha f(v)+(1-\alpha)f(u)$$

such a function is called pre-invex function with respect to $\eta(v,u)$ and the property "for each $u,v \in K$, $\alpha \in [0,1]$, $\alpha+\eta(v,u) \in K$" is called η-connectedness.

Now we define an unconstrained optimization problem as follows:

(P_1) Minimize $f(u)$, $\forall\ u \in K$, where K is a subset of n-dimensional enclidean space R^n, $f:K \to R$.

Now we state the following theorem, which shows equivalence between variational like inequality with problem (P_1).

THEOREM 5.2.5 [93]. Let K be a subset in R^n having η-cnnectedness, $u_o \in K$, and f be differentiable at u_o and $T(u_o) = \nabla f(u_o)$. Then the two statements holds good.

(I) If u_o is a minimum of problem (P_1), then u_o is a solution of variational-like inequality (5.2.1)

(II) If f is a invex function with respect to $\eta(v,u)$, u_o is a solution of (5.2.1), then u_o is a minimum of problem (P_1).

5.3. EXISTENCE IN REFLEXIVE BANACH SPACES.

Let E be reflexive Banach space and K be a non-empty closed convex subset of E. We denote 2^E, the set of all non empty subsets of E and by Conv (A), for all $A \subset E$ the convex hull of A.

DEFINITION 5.3.1. A mapping $T:E \to E^*$ is said to be η-monotone if there exists a continuous mapping $\eta:K \times K \to E$, such that

$$\langle T(u)-T(v),\eta(u,v)\rangle \geq 0, \text{ for all } u,v \in K$$

It is well known that Minty lemma [51] plays an important role in the theory of variational inequalities, the extension of this lemma for variational-like inequalities is as follows:

LEMMA 5.3.1 [85]. Let $T:K \to E^*$ be η-monotone and hemicontinuous mapping on K. Also, let $\eta:K \times K \to E$ be continuous mapping which is linear in the first argument and $\langle T(u),\eta(u,u)\rangle = 0$, for all $u \in K$, then u satisfies

$$u \in K : \langle T(u),\eta(u,v)\rangle \geq 0, \quad \text{for all} \quad v \in K \qquad (5.3.1)$$

If and only if satisfies

$$u \in E : \langle T(v),\eta(v,u)\rangle \geq 0, \quad \text{for all} \quad v \in K \qquad (5.3.2)$$

Now we have the following existence result:

THEOREM 5.3.1 [85]. Let E be a reflexive Banach space and let K be nonempty bounded closed convex subset of E such that

1. $T:K \to E^*$ is η-monotone and hemicontinuous on K,
2. $\eta:K \times K \to E$ is continuous mapping which is linear in the first argument.
3. $\langle T(u),\eta(u,u)\rangle = 0$, for all $u \in K$

Then there exists $u_o \in K$, such that

$$\langle T(u_o), \eta(u,u_o)\rangle \geq 0, \quad \text{for all } u \in K.$$

5.4. EXISTENCE THEORY IN HAUSDORFF TOPOLOGICAL VECTOR SPACE

Let E be Hausdorff topological vector space. $\langle E,E^*\rangle$ is dual system of locally convex spaces, K is a closed convex subset of E and D is a nonempty compact convex subset of K then we have the following result.

THEOREM 5.4.1 [85]. Assume that

(1^o). $T:K \to E^*$ is continuous

(2^o), $\eta:K \times K \to E$ is continuous

(3^o). There exist a real valued function $h:K \times K \to R$ such that

(I) $\langle T(v),\eta(u,v)\rangle + h(u,v) \geq 0$, for all $(u,v) \in K \times K$,

(II) the set $\{u \in K : h(u,v) > 0\}$ is convex for every $v \in K$

(III) $h(u,u) \leq 0$, for every $u \in K$,

(IV) There exist a nonempty compact convex subset D of K such that fvor every $v \in K \setminus D$, there exist a point $u \in D$ with

$$\langle T(v),\eta(u,v)\rangle < 0$$

Then the variational-like inequality (5.2.1) has a solution.

PROOF. For each $u \in K$, we denote

$$D(u) = \{v \in D : \langle T(v), \eta(u,v)\rangle \geq 0\}$$

and from assumption ($\mathring{1}$) and ($\mathring{2}$) we have $D(u)$ is closed in D.

We know that every element $u_o \in \bigcap_{u\in K} D(u)$ is a solution of the problem (5.2.1), we prove that $\bigcap_{u\in K} D(u) \neq \phi$.

Since D is compact it is sufficient to show that family $\{D(u)\}_{u\in K}$ has a finite inter section property.

Let $u_1, u_2, \ldots u_m \in K$ be given

We put $A = \mathrm{Conv}(D \cap \{u_1, u_2, \ldots, u_m\})$ and we have that A is compact convex subset of K.

We consider the following multivalued mappings:

$F_1(u) = \{v \in A : \langle T(v), \eta(u,v)\rangle \geq 0\}$, for every $u \in K$

and

$F_2(u) = \{v \in A : h(u,v) \leq 0\}$, for every $u \in K$

Since bilinear form $\langle .,. \rangle$ is continuous, from assumption ($\mathring{1}$) and ($\mathring{2}$), it follows that $F_1(u)$ is closed subset of a compact convex set A. Hence $F_1(u)$ is compact. Also from assumptions $\mathring{3}$(i) and $\mathring{3}$(iii), $F_1(u)$ is nonempty.

Now we prove that F_2 is a KKM-mapping. Indeed, if we suppose that there exists $x_1, x_2, \ldots, x_n \in A$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$$\sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{J=1}^{n} F_2(x_j)$$

then we have

$$h(x_j, \sum_{i=1}^{n} \alpha_i x_i) > 0, \quad \text{for} \quad 1 \leq J \leq n$$

By assumption 3°(ii), we have

$$h(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i x_i) > 0$$

which is contradiction to assumption 3°(iii). Therefore F_2 is a KKM-mapping.

Since from assumption 3°(i), we have $F_2(u) \subset F_1(u)$, for every $u \in K$, we obtain that F_1 is also a KKM-mapping.

Applying Theorem (1.2.1) to F_1 we get $\bigcap_{u \in A} F_1(u) \neq \Phi$, that is, there exists a point $v_o \in A$, such that

$$\langle T(v_o), \eta(u, v_o) \rangle \geq 0, \text{ for all } u \in A$$

By assumption 3°(iv), we have that $v_o \in D$ and moreover $v_o \in D(u_i)$, for every $1 \leq i \leq m$.

Hence $\{D(u)\}_{u \in K}$ has a finite intersection property.

COROLLARY 5.4.1. Assume that assumption (1°), (2°) and (3°)(iv) of theorem (5.4.1) are satisfied. Also suppose that

$$\langle T(u), \eta(u,u)\rangle \geq 0, \text{ for all } u \in K.$$

Then the variational-like inequality problem (5.2.1) has a solution.

5.5. EXISTENCE THEORY WITHOUT CONVEXITY.

In this section we study the existence theorem for variational-like inequalities, by replacing convexity assumptions with merely topological properties.

Now we state and prove the main theorem of this section as follows:

THEOREM 5.5.1[85]. Let $(E, \{\Gamma_A\})$ be an H-Banach space, $T: E \to E^*$ and $\eta(.,.) : K \times K \to E$ be continuous. Assume that

1°. for each $v \in E$, $B_v = \{x \in E : \langle T(v), \eta(u,v) < 0\}$ is H-convex or empty.

2°. $\langle T(u), \eta(u,u)\rangle \geq 0$, for all $u \in E$,

3°. there exist a compact set $L \subset E$ and an H-compact set $K \subset E$, such that for every weakly H-convex set D with $K \subset D \subset E$

$$\{v \in D : \langle T(v), \eta(u,v)\rangle \geq 0, \text{ for all } u \in D\} \subset L$$

Then there exists an u_o such that

$$\langle T(u_o), \eta(u,u_o)\rangle \geq 0 \quad \text{for all } u \in E.$$

PROOF. Let

$$F(u) = \{v \in E : \langle T(v), \eta(u,v)\rangle \geq 0\}, \quad u \in E$$

We prove that F is a H-KKM-mapping. Suppose that F is not an H-KKM mapping. Then there exists a finite subset A of E such that

$$\Gamma_A \not\subset \bigcup_{u \in A} F(u)$$

Thus there exists $z \in \Gamma_A$ such that

$$z \notin F(u), \text{ for all } u \in A$$

that is

$$\langle T(z), \eta(u,z)\rangle < 0, \text{ for all } u \in A.$$

By assumption (1°), $A \subset B_z$ and $\Gamma_A \subset \Gamma_{B_z}$, since B_z is H-convex. Therefore $z \in B_z$,

$$<T(z),\eta(z,z)> < 0,$$

which contradicts assumption 2°. Thus $\Gamma_A \quad \bigcup_{u\varepsilon A} F(u)$ for every finite subset A of E. Hence F is an H-KKM mapping.

Since $T,\eta(.,.)$ and $<.,.>$ are continuous we have $F(u)$ is closed for every $u \varepsilon E$, that is condition ($\overset{\circ}{a}$) of theorem (1.2.3) holds.

It is easy to see that the assumption (3°) of this theorem and condition (b°) of Theorem (1.2.3) are the same. Thus we have

$$\bigcap_{u\varepsilon E} F(u) \neq \phi.$$

Hence there exists $u_o \varepsilon E$ such that

$$<T(u_o),\eta(u,u_o)> \geq 0 \quad \text{for all} \quad u \varepsilon E.$$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] ADAMS, R.A. : Sobolev Spaces, Academic Press, New York (1975).

[2] AHMAD, R. : Numerical Analysis of Variational Inequalities in Modelling of Contact Problems, Ph.D., Thesis, A.M.U., Aligarh, India, 1992.

[3] ANSARI, Q.H.: On Certain Problems concerning variational inequalities, Ph.D. Thesis, A.M.U., Aligarh, India, 1988.

[4] AUBIN, J.P. : "Approximation of variational inequalities", Functional Analysis and optimization (edited by E.R. Cainiello), Academic Press, New York, 1966, pp. 7-14.

[5] AVRIEL, M. : "Nonlinear Programming Theory: Theory and methods", Prentice Hall, Englewood, Cliffs, N.J., 1976.

[6] BAIOCCHI, C. and CAPELO, A.: Variational and Quasivariational Inequalities, John Wiley and Sons, New York, 1984.

[7] BAIOCCHI, C., GASTALDI, F. and TOMARELLI, F.: Some existence results on noncoercive variational inequalities, Ann. Senola Norm sup Pisa Cl. Sci., 13 (1986), 617-695.

[8] BARDARO, C. and CAPITELLI, R.: Some further generalizations of the Knoster-Kuratowski-Mazukiewicz Theorems and Minimax Inequalities, J. Math. Anal. Appl. 132 (1988), 484-490.

[9] BEN-ISRAEL, A. and MOND, B.: What is invexity, J. Austral. Math. Soc., 28 (1986), 1-9.

[10] BRÉZIS, H. and STAMPACCHIA, G.: "Sur la regularite de la solution d'inequations elliptiques", Bull. Soc., Math. France, 96 (1968), 153-180.

[11] BREZZI, F., HAGER, W. and RAVIART, P.A.: Error estimation of Finite element solution of variational Inequalities, Part I: Primal Theory, Numer. Math., 28 (1977), 431-433.

[12] CHAN, D. and PANG, S.: The generalized quasivariational inequality problem", Math. Oper. Res. 7 (1982) 211-222.

[13] CHEN, G.Y. and CHENG, G.M.: Vector Variational Inequality and Vector Optimization, Lecture Notes in Economics and Mathematical System, Springer Verlag, Heidelberg, Germany, 285 (1987).

[14] CHEN, G.Y. and CRAVEN: A vector Variational Inequality and Optimization over an efficient set, Zeitschrift fur operations Research, 3(1990),1-12.

[15] CHEN, G.Y. and YANG, X.Q.: Vector Complimentarity Problem and its Equivalence with weak Minimal Element in Ordered Banach Space, Jr. Math.Anal. Appl. 153 (1990), 136-158.

[16] CHEN, G.Y. : Existence of solution for Vector Variational Inequality: An Extension of the Hartman-Stampacchia Theorem, JOTA, 74(1992),445-456.

[17] CRAVEN, B.D. : Invex function and constrained local minima, Bull. Austral. Math. Soc., Ser.B, 24 (1981), 357-366.

[18] CRAVEN, B.D. : Invex function and duality, J. Austral. Math. Soc. Ser.A, 39 (1985), 1-20.

[19] CIARLET, P.G.: The Finite element Method for Elliptic Problems North Holland, 1987.

[20] COTTLE, R.W., GIANESSI, F. and LIONS, J.L.: Variational Inequality and Complementarity Problems, John Wiley and Sons, New York, 1980.

[21] CRANK, J. : Free and Moving Boundary Value Problems, Claredon Press, Oxford, England, 1984.

[22] DUGUDJI, J. and GRANAS, A.: KKM-Maps and Variational Inequalities, Annali dell Scuola Normale Superiore, Pisa, Italy, (1989), 679-682.

[23] DUVAUT, G. and LIONS, J.L.: Inequalities in Mechanics and Physics, Springer-Verlag, Berlin, 1976.

[24] FALK, R.S. : "Error estimation for the approximation of a class of variational inequalities", Math. Comp. 28 (1974), 963-971.

[25] FAN, K. : A Generalization of Tychonoff's Fixed-Point Theorem, Math. Anals, 142 (1961), 305-310.

[26] FAN, K. : A Minimax Inequality and Applications "Inequalities" III O. Shisha ed. Academic Press, York, (1972), 103-113.

[27] FERRO, O. : Theorems of the Alternative for set-valued Functions in Infinite Dimensional Spaces, Optimization, 20 (1989), 167-175.

[28] FICHERA, G. : Problemi elastostalici con vincoli unilaterli: il problema de Signorini con ambigue condizioni al contono, Atti. Acad. Naz. Lincei, Mem. Cl. Sci. Fiz. Mat. Nat. Sez. Ia., 7(8) (1964), 91-140 (1964).

[29] FRIEDMAN, A. : Variational Principles and Free Boundary Problems, John Wiley, New York, 1982.

[30] GIANESSI, F. : Theorem's of Alternative, Quadratic Programs and Complementarity Problem's, Variational Inequality and Complementarity Problems, Edited by R.W. Cottle, F. Gianessi and J.L. Lions, John Wiley and Sons Clichester, England (1980) 151-186.

[31] GLOWINSKI, R., LIONS, J.L. and TREMOLIERS, R.: Numerical Analysis of Variational Inequalities, North-Holland Amsterdam, Holland, 1981.

[32] GLOWINSKI, R.: Numerical Methods for Nonlinear Variational Problems, Physics Series, Springer-Verlag, New York, 1984.

[33] HANSON, M. : On sufficiency of Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981), 545-550.

[34] HANSON, M. and MOND, B.: Necessary and sufficient conditions in constrained optimization, Math. Programming 37 (1987), 51-58.

[35] HARKER, P.T. and PANG, J.G.: Finite dimensional Variational Inequalities and Nonlinear Complimentarity problem; A Survey of Theory, Algorithms and Applications, Math. Programming, 48 (1990), 161-170.

[36] HARTMAN, P. and STAMPACCHIA, G.: On some nonlinear elliptic differential functional equations, Acta Math. 115 (1966), 217-310.

[37] HOVARTH, C. : Some results on multivalued mappings and inequalities, without convexity in "Nonlinear and Convex Analysis", Lecture Notes in Pure and App. Math. Series, Springer-Verlag, 107 (1987).

[38] HOLMES, R.B. : Geometric Functional Analysis and its Appplications, Springer-Verlag, Heidelberg, Germany 1975.

[39] ISAC, G. : "On the Implicit Complementarity Problem in Hilbert Space", Bull. Austral. Math. Soc., 32 (1985), 251-260.

[40] ISAC, G. : A special Variational Inequality and the Implicit Complementarity Problem, J. Fac. Sc. Univ. of Tokyo, 37 (1990), 109-127.

[41] ISTRATESCU, V.I.: Fixed Point Theory, D. Riedel, 1981.

[42] JAMENSON, G. : Ordered Linear Spaces, Springer-Verlag, Heidelberg, Germany, 1970.

[43] KAKUTANI, S. : "A generalization of Brouwer's fixed point theorem, Duke, Math. J. 8 (1941).

[44] KARAMARDIAN, S.: Generalized Complementarity Problem, JOTA, 8 (1971).

[45] KIKUCHI, N. and ODEN, J.T.: Contact Problems in Elasticity, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1988.

[46] KINDERLEHRER, D. and STAMPACCHIA, G.: An introduction to variational inequalities and their applications, Academic Press, New York, 1980.

[47] LEWY, H. and STAMPACCHIA, G.: "On the regularity of the solution of a variational inequality", Comm. Pure. Appl. Math. 22 (1969), 153-188.

[48] LIONS, J.L. and STAMPACCHIA, G.: Variational Inequality, Comm. Pure. Appl. Math. 20 (1967), 493-517.

[49] LIONS, J.L. and MAGENES, E.: Non-homogeneous BVP's and Application I, Springer-Verlag, Berlin (1972).

[50] MARTEIN, L. : Stationary Points and Necessary Conditions in Vector Extremum Problem's, Research Report No.133, Optimization and Operations Research Group, Deptt. of Math., Univ. of Pisa, Pisa Italy, 1986.

[51] MINTY, G.J. : Monotone (Nonlinear) Operators in Hilbert Space, Duke. Math. J. 1 (1976), 260-266.

[52] MOND, B. and HANSON, M.A.: "On duality with generalized convexity, Math. Operation forch, Statist. Ser. Optim. 15 (1984), 313-317.

[53] MOORE, J.J. : Coercivity Conditions in nonlinear complimentarity problems, SIAM, Rev. 16 (1974), 1-16.

[54] MOSCO, U. : "Convergence of Convex sets and of solutions of variational inequalities", Advances in Math. 3 (1969), 510-585.

[55] MOSCO, U. : "Approximation of the solutions of variational inequalities", Ann. Senola Norm. Sup. Pisa (3) 21 (1976), 337-394.

[56] MOSCO, U. : "An Introduction to the approximate solution of variational inequalities", Instituto Mathematico G. Castelnuovo, Universila degli Studi di Roma, (1977-78).

[57] MOSCO, U. : Some Introductory Remarks on Implicit Variational Problems, Proceeding Regional Workshop supported by ICTP, Trieste, Italy, NBHM, UGC-CSIR, UNESCO, March 6-16 (1989), Aligarh Muslim University, Aligarh, India (To be published by Mc Millan India).

[58] MOSCO, U. : Some Introductory remarks on implicit variational inequalities, (Editor A.H. Siddiqi) Proceedings of the Regional Workshop on Applicable Mathematics, supported by ICTP, Trieste, Italy, NBHM, CSIR, Aligarh, India, 1993.

[59] MOSCO, U. and STRAG, G.: One sided approximation and variational approximation, Bull. Amer. Math. Soc. B (1974), 308-312.

[60] NITSCHE, J. : "Lineare Spline-Funktionen and die Methoden von Ritz fur elliptische Randwert problem" Arch. Rational Mech. Anal. 36 (1970), 348-355.

[61] NOOR, M.A. : L_2-Estimates for variational inequalities, C.R. Math. Rep. Acad. Canada, Vol.IV, No.3, June 1982.

[62] NOOR, M.A. : "General Nonlinear Variational Inequalities" J. Math. Anal. Appl. 126 (1987), 78-84.

[63] NOOR, M.A. : "On a class of variational inequalities", J. Math. Anal. Appl. 128 (1987), 138-155.

[64] NOOR, M.A. : General Variational Inequalities, Appl. Math. Letters 1 (1988), 119-122.

[65] NOOR, M.A. : Iterative Algorithm for Quasivariational Inequalities, Pan American Mathematical Journal, 2 (1992), 12-26.

[66] NOOR , M.A. : General Algorithm and Sensitivity Analysis for Variational Inequalities, Jr. of Appl. Math. and Stochastic Anal, 5 (1992), 29-42.

[67] NOOR, M.A. : Wiener-Hopf Equation and Variational Inequalities, JOTA, Vol.79, No.1, October (1993).

[68] NOOR, M.A. : General Nonlinear Complimentarity Problem, G. Bernhard Riemann: A mathematical Legacy, Edited by Rassias, T.M. and Srivastav, H.M. Hadronic Press, Palm Harbor, Floriada, 1994.

[69] PARIDA, J., SAHOO, M. and KUMAR, A.: A Variational-like Inequality Problem, Bull. Austral. Math. Soc. 39 (1989), 225-231.

[70] PITONYAK, A., SHI, P. and SHILLOR, M.: On an iterative Method for variational inequalities, Numerische Mathematische, 58 (1990), 231-242.

[71] RUEDA, N.G. and HANSON, M.A.: Optimality Criteria for Mathematical Programming involving generalized Invexity, J. Math. Anal. Appl. 130 (1988), 375-385.

[72] SHI, P. : An Iterative Method for Obstacle Problems via Green's Function, Nonlinear Analysis: Theory, Methods and Applications, 15 (1990), 339-344.

[73] SHI, P. : Equivalence of Variational Inequalities with Wiener-Hopf Equations, Proceedings, American Mathematical Society, Vol.III, Number 2, Feb. (1991).

[74] SIDDIQI, A.H.: Functional Analysis with Application, Tata McGraw Hill Publishing Company, New Delhi (1989).

[75] SIDDIQI, A.H: Lecture Notes on Multivalued Mappings, Regional Workshop on Applicable Mathematics, AMU, Aligarh 6-16, March 1989, supported by ICTP, Trieste, Italy, NBHM, CSIR, India, McMillan (to appear).

[76] SIDDIQI, A.H.:Some Remark on Variational and Quasivariational Inequalities of Monotone Operators (Preprint).

[77] SIDDIQI, A.H: On a Class of General Variational Inequalities (Preprint).

[78] SIDDIQI, A.H: Introduction to Variational Inequalities, in (Editor A.H. Siddiqi) Proceedings of the Regional Workshop on Applicable Mathematics, supported by ICTP, Trieste, Italy, NBHM, CSIR Aligarh, India, 1993.

[79] SIDDIQI, A.H: Some recent development in variational inequality Invited lecture international conference on Topological Algebra and related topics in memory of Prof. Prof. S.I. Husain, May 2-6, McMaster University

[80] SIDDIQI, A.H. and ANSARI, Q.H.: An Iterative Method for Generalized Variational Inequalities, Math. Japonica 34 (1989), 475-481.

[81] SIDDIQI, A.H. and ANSARI, Q.H. : An Algorithm for a Class of Quasivariational Inequalities, J. Math. Anal. Appl. 145 (1990), 413-418.

[82] SIDDIQI, A.H. and ANSARI, Q.H. : Strongly Nonlinear Quasi-variational Inequalities, J. Math. Anal. Appl. 149 (1990), 444-450.

[83] SIDDIQI, A.H. and ANSARI, Q.H.: General Strongly Nonlinear Variational Inequalities, J. Math. Anal. Appl. 149 (1990), 413-418.

[84] SIDDIQI, A.H. and ANSARI, Q.H.: On the Nonlinear Implicit complementarity Problem, International J. Math. Sci. 15 (1993), 783-790.

[85] SIDDIQI, A.H., KHALIQUE, A. and ANSARI, Q.H.: On Variational like Inequalities, Annales des Sci. Math. du Qubec 18 (1994), 95-104.

[86] SIDDIQI, A.H., ANSARI, Q.H. and KAZMI, K.R.: On Nonlinear Variational Inequalities, to appear in Indian Jr. Pure & Appl. Math. (1994).

[87] SIDDIQI, A.H., ANSARI, Q.H. and KHALIQUE, A.: On General Vector Variational Inequality, to appear in JOTA (1995).

[88] SIDDIQI, A.H., ANSARI, Q.H. and AHMAD, R.: On Vector Variational Inequalities, (Submitted).

[89] SIDDIQI, A.H., ANSARI, Q.H. and AHMAD, R.: Some Remarks on Variational-like Inequalities, (Submitted).

[90] SIDDIQI, A.H., KHALIQUE, A. and ANSARI, Q.H.: On General Nonlinear Variational Inequalities (Submitted).

[91] STAMPACCHIA, G.: Formes bilinears coercivitives sur les ensembles convexes, C.R. Acad. Sci., Paris, 285 (1964), 4413-4416.

[92] SPECK, F.O. : General Weiner-Hopf factorization methods Research Notes in Math., No.119, Pitman, Advanced Publishing Program (1985).

[93] YANG, X.Q. and GUANG-YA CHEN: A Class of Nonconvex Functions and Pre-Variational Inequalities, J. Math. Anal. Appl. 169 (1992), 359-373.